Peano phenomenon for stochastic equations with local time.

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Abstract

In this paper we investigate weak convergence of measures generated by solutions of stochastic equations with local time and small diffusion while the last one tends to zero. In case the correspondent ordinary differential equation has infinitely many solutions we prove that limit measure concentrated with some weights on its extreme solutions. Formulae for weights are obtained.

Keywords: Peano phenomenon, stochastic equation, local time, weak convergence of measures.

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1 Introduction.

The problem of convergence of measures generated by solutions of stochastic differential equations (SDEs) with small diffusion

$$x_\varepsilon(t) = x_0 + \int_0^t b(x_\varepsilon(s))ds + \varepsilon w(t),$$

as $\varepsilon \to 0$ to the measure concentrated on the unique solution of the ordinary differential equation (ODE)

$$x'(t) = b(x(t)), x(0) = x_0,$$

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was considered in several papers. Mention books by Stroock-Varadhan [1] and Wentzell-Freidlin [6]. Non-uniqueness case of solutions of ODE (Peano phenomenon) also was considered. Mention such authors as Baldi [3], Baldi-Bafico [4], Veretennikov [5], Gradinaru-Herrmann-Roynette [8], Buckdahn-Quincampoix-Ouknine [11], Krykun-Makhno [12].

In this paper we consider SDE with local time, continuous drift and small diffusion

$$\xi_\varepsilon(t) = \beta L^{\xi_\varepsilon}(t, 0) + \int_0^t b(\xi_\varepsilon(s))ds + \varepsilon \int_0^t \sigma(\xi_\varepsilon(s))dw(s), \qquad t \in [0, 1]. \quad (1)$$

With this equation we connect Cauchy problem for ODE

$$y'(t) = b(y(t)), \quad y(0) = 0. \tag{2}$$

and we prove weak convergence of measures generated by solutions of (1) to measure concentrated with some weights on extreme solutions of (2). Formulae for calculating their weights are obtained.

We use the following notation: $I_A(x)$ be the indicator function of the set A; $a^+ = \max(a, 0)$; $\mathbb{C}[0, \infty)-$ be a space of continuous functions $f(t)$, $t \in [0, \infty)$, with the metric of uniform convergence on compact subsets of $[0, \infty)$:

$$\rho(f, g) = \sum_{N=1}^{\infty} \frac{1}{2^N} \frac{\sup_{t \in [0,N]} |f(t) - g(t)|}{1 + \sup_{t \in [0,N]} |f(t) - g(t)|}.$$

Let $\mathfrak{B}$ be $\sigma-$ algebra of Borel sets of this space. Denote $f(x) \sim g(x)$ as $x \to x_0$, if equality

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = 1$$

takes place.

Let $(\Omega, \mathfrak{F}, \mathfrak{F}_t, \mathsf{P})$ be the probability space with a flow of σ-algebras $\mathfrak{F}_t$, $t \geq 0$, $(w(t), \mathfrak{F}_t)$ be standard one-dimensional Wiener process.

The function $\mathrm{sgn}(x)$ is defined as:

$$\mathrm{sgn}(x) = \begin{cases} 1, \text{ if } x > 0, \\ 0, \text{ if } x = 0, \\ -1, \text{ if } x < 0. \end{cases}$$

Definition 1. [7, Definition 4.7(1)] Equation (1) has a **weak solution** if for given functions $b(x)$, $\sigma(x)$ and a constant β there exist a probability space

$(\Omega, \mathfrak{F}, \mathfrak{F}_t, \mathsf{P})$ with flow of σ-algebras $\mathfrak{F}_t$, $t \geq 0$, continuous semimartingale $(\xi(t), \mathfrak{F}_t)$ and a standard one-dimensional Wiener process $(w(t), \mathfrak{F}_t)$ such as

$$L^{\xi}(t,0) = \lim_{\delta \to 0} \frac{1}{2\delta} \int_0^t I_{(-\delta,\delta)}(\xi(s))ds \qquad (3)$$

exists almost surely and (1) takes place almost surely.

Definition 2. [7, Definition 4.7(2)] Equation (1) has a **strong solution** if for the given functions $b(x)$, $\sigma(x)$ and a constant β relations (1) and (3) take place almost surely (a.s.) on the given probability space $(\Omega, \mathfrak{F}, \mathfrak{F}_t, \mathsf{P})$ with flow of σ-algebras $\mathfrak{F}_t$, $t \geq 0$ and given Wiener process $(w(t), \mathfrak{F}_t)$.

Definition 3. [2, Chapter IV, Definition 1.4] The **uniqueness of solutions** (or **weak uniqueness** or **uniqueness in the sense of probability**) for equation (1) holds if whenever X and X' are two solutions of (1) such as $X(0) = 0$ a.s. and $X'(0) = 0$ a.s., then the laws on the space $\mathbb{C}([0, \infty])$ of the processes X and X' coincide.

Definition 4. [2, Chapter IV, Definition 1.5] We say that the **pathwise uniqueness** (or **strong uniqueness**) **of solutions** for equation (1) holds if whenever X and X' are any two solutions defined on the same probability space $(\Omega, \mathfrak{F}, \mathfrak{F}_t, \mathsf{P})$ with the same flow of σ-algebras $\mathfrak{F}_t$, and the same one-dimensional Wiener process $(w(t), \mathfrak{F}_t)$ such as $X(0) = X'(0)$ a.s., then $X(t) = X'(t)$ for all $t > 0$ a.s.

For coefficients of equation (1) we consider next condition (I).

Condition (I):

I_1. Function $b(x)$ be continuous, $b(0) = 0$ and null point is its unique zero.

I_2. There is a constant $\Lambda \geq 1$ such as

$$|b(x)|^2 + \sigma^2(x) \leq \Lambda(1 + |x|^2), \ \ \sigma^2(x) \geq \Lambda^{-1}.$$

I_3. For every x and y the product $\sigma(x)\sigma(y) > 0$ and for every $N < \infty$

$$\sup_{-N = x_0 < ... < x_k = N} \sum_{i=1}^{k} |\sigma(x_i) - \sigma(x_{i-1})| < \infty.$$

I_4. The constant $|\beta| < 1$.

The paper is organized as follows. Section 2 contains results for ordinary differential equations. Main results of the paper are formulated in Section 3 and are proved in Section 4. Section 5 contains some examples.

2 Ordinary Differential Equations.

Consider Cauchy problem (2). By conditions I_1 and I_2 for the function $b(x)$ this problem has at least one solution – zero – and all the solutions of this problem pass through the point $(0;0)$. This set of all curves is called integral funnel and denote by $\mathfrak{R}$. From existence of two different solutions it follows that there are infinitely many solutions. Each solution from the integral funnel is located between two special solutions which are called extremal - upper $\overline{y}(t)$ and lower $\underline{y}(t)$, where $\overline{y}(t) = \sup\{y(t), y(t) \in \mathfrak{R}\}$, $\underline{y}(t) = \inf\{y(t), y(t) \in \mathfrak{R}\}$.

Note that if $b(x)x < 0$ for $x \neq 0$, then the problem (2) has only zero solution.

Existence of nonzero solution of (2) requires convergence at least one of integrals

$$\int_0^\delta \frac{1}{b(y)}dy, \quad \int_{-\delta}^0 \frac{1}{b(y)}dy. \tag{4}$$

So interesting are the following cases:

A_1. Function $b(x)x > 0$ for $x \neq 0$ and both integrals in (4) are convergent.
A_2. Function $b(x)x > 0$ for $x \neq 0$ and the first integral in (4) converges and the second one diverges.
A_3. Function $b(x)x > 0$ for $x \neq 0$ and the first integral in (4) diverges and the second one converges.
A_4. Function $b(x) > 0$ for $x \neq 0$ and the first integral in (4) converges.
A_5. Function $b(x) < 0$ for $x \neq 0$ and the second integral in (4) converges.

Let $H(x) = \int_0^x \frac{1}{b(y)}dy$ for $x \geq 0$ and $K(x) = \int_x^0 \frac{1}{b(y)}dy$ for $x \leq 0$. If condition I_1 takes place, then these functions are strictly monotone. Denote the inverse functions of $H(x)$ and $K(x)$ by $H^{-1}(x)$ and $K^{-1}(x)$ respectively.

Lemma 1. [12, Lemma 2.2 and Lemma 2.3].
1. *In case A_1 all nonzero solutions of the problem (2) are given by:*

$$y_\lambda(t) = H^{-1}\big((t - \lambda)^+\big), \quad \lambda \geq 0, \tag{5}$$

$$y_\mu(t) = K^{-1}\big(-(t - \mu)^+\big), \quad \mu \geq 0. \tag{6}$$

In this case the extremal solutions are $\overline{y}(t) = H^{-1}(t)$, $\underline{y}(t) = K^{-1}(t)$.
2. *In cases A_2 and A_4 all nonzero solutions of the problem (2) have the form*

4

(5) and the extremal solutions are $\overline{y}(t) = H^{-1}(t)$, $\underline{y}(t) = 0$.

3. In cases A_3 and A_5 all nonzero solutions of the problem (2) have the form (6) and the extremal solutions are $\overline{y}(t) = 0$, $\underline{y}(t) = K^{-1}(t)$.

For study of the weights of a limit measure we need to compute an expression

$$\Gamma_K = \lim_{\varepsilon \to 0} \frac{-A_\beta^\varepsilon(-K)}{A_\beta^\varepsilon(K) - A_\beta^\varepsilon(-K)}, \tag{7}$$

where $\quad A_\beta^\varepsilon(x) = \int_0^x \exp\left\{ -\frac{2}{\varepsilon^2} \int_0^z \frac{(1 + \beta\operatorname{sgn}(v))b((1 + \beta\operatorname{sgn}(v))v)}{\sigma^2((1 + \beta\operatorname{sgn}(v))v)} dv \right\} dz.$

To calculate Γ_K put

$$L(x) = \int_0^x \frac{b(y)}{\sigma^2(y)} dy. \tag{8}$$

Lemma 2. Let $b(x)x > 0$ for $x \neq 0$, for some constants $d > 0$, $\delta > 0$ and γ as $x \to 0+$

$$L(x)|\ln L(x)|^\gamma \sim dx^\delta \tag{9}$$

and for some constants $k > 0$, $\mu > 0$ and θ as $x \to 0-$

$$L(x)|\ln L(x)|^\theta \sim k|x|^\mu. \tag{10}$$

Then the value of Γ_K is independent on K (so we denote it by Γ) and take place the following statements:

1. If $\delta = \mu$ and $\gamma = \theta$, then

$$\Gamma = \frac{1}{1 + \frac{1-\beta}{1+\beta}\left(\frac{k}{d}\right)^{\frac{1}{\delta}}}.$$

2. If $\delta < \mu$ or $\delta = \mu$ and $\gamma < \theta$, then $\Gamma = 1$.
3. If $\delta > \mu$ or $\delta = \mu$ and $\gamma > \theta$, then $\Gamma = 0$.

Proof. Denote

$$L_\beta(x) = \int_0^x \frac{(1 + \beta\operatorname{sgn}(y))b((1 + \beta\operatorname{sgn}(y))y)}{\sigma^2((1 + \beta\operatorname{sgn}(y))y)} dy.$$

Let $x > 0$ then:

$$L_\beta(x) = \int_0^x \frac{(1 + \beta)b((1 + \beta)y)}{a((1 + \beta)y)} dy = \int_0^{(1+\beta)x} \frac{b(y)}{a(y)} dy = L((1 + \beta)x),$$

5

where function $L(x)$ defined by (8). So from the condition (9) we have

$$L_\beta(x)|\ln L_\beta(x)|^\gamma \sim d(1+\beta)^\delta x^\delta = d^* x^\delta,$$

where $d^* = d(1+\beta)^\delta$.

For $x < 0$ from the condition (10) by analogy we have

$$L_\beta(x)|\ln L_\beta(x)|^\gamma \sim k^*|x|^\mu,$$

where $k^* = k(1-\beta)^\mu$.

Using now [12, lemma 2.8] we get a statement of the lemma 2.

Lemma 2 is proved.

3   Main results.

It is known that if conditions I_2 and I_4 are hold, then there exists a weak unique weak solution of equation (1) [7, theorem 4.35]. By connection between stochastic equations with local time and Ito stochastic equations [9], one can prove next theorem.

Theorem 1. *Let for the coefficients of the equation (1) conditions I_2, I_3 and I_4 are hold. Then for every fixed $\varepsilon > 0$ equation (1) has a pathwise unique strong solution.*

Consider the stochastic equation with local time (1) and corresponding to it Cauchy problem (2). Denote by μ_ε measures generated by processes $\xi_\varepsilon(\cdot)$ on the space $(\mathbb{C}[0,\infty), \mathfrak{B})$.

Theorem 2. *Suppose that the conditions I_1, I_2, I_4, A_1, (9), (10) are hold for the coefficients of equation (1). Then for measures $\{\mu_\varepsilon\}$ and for any bounded continuous functional F defined on the space $\mathbb{C}[0,\infty)$, the equality*

$$\lim_{\varepsilon \to 0} \int_{\mathbb{C}[0,\infty)} F(f)\mu_\varepsilon(df) = \Gamma F(\overline{y}) + (1-\Gamma)F(\underline{y}), \qquad (11)$$

takes place, where $\overline{y}, \underline{y}$ are extremal solutions of the problem (2), and the value of Γ defined by the lemma 2.

For the investigation of cases $A_2 - A_5$ we will use a comparison theorem. So we need a pathwise unique strong solutions of SDEs.

Theorem 3. *Suppose that for the coefficients of the equation (1) the condition (I) holds.*

In the cases A_2, A_4 and if condition (9) takes place, then limit measure for the sequence $\{\mu_\varepsilon\}$ is concentrated on the upper extremal solution of the problem (2).

In the cases A_3, A_5 and if condition (10) takes place, then limit measure for the sequence $\{\mu_\varepsilon\}$ is concentrated on the lower extremal solution of the problem (2).

4 Proof of theorems.

Proof of theorem 1.

Solution of equation (1) is strongly connected with solution of Ito's equation. Denote

$$\kappa(x) = \begin{cases} (1-\beta)x, & x \leq 0, \\ (1+\beta)x, & x \geq 0, \end{cases} \tag{12}$$

and let $\quad \varphi(x) = \begin{cases} \dfrac{x}{1-\beta}, & x \leq 0 \\ \dfrac{x}{1+\beta}, & x \geq 0 \end{cases}\quad$ be inverse function of $\kappa(x)$. Further set

$$\widetilde{b}(x) = \frac{b(\kappa(x))}{1+\beta\mathrm{sgn}(x)}, \quad \widetilde{\sigma}(x) = \frac{\sigma(\kappa(x))}{1+\beta\mathrm{sgn}(x)},$$

and consider Ito's stochastic equation:

$$\eta_\varepsilon(t) = \int_0^t \widetilde{b}(\eta_\varepsilon(s))ds + \varepsilon \int_0^t \widetilde{\sigma}(\eta_\varepsilon(s))dw(s), \qquad t \in [0,1]. \tag{13}$$

For function $\widetilde{b}(t)$, $\widetilde{\sigma}(t)$ also take place conditions I_2 and I_3 as for functions $b(t)$, $\sigma(t)$. Consequently we can use [12, theorem 3.2] for equation (13). So for every fixed $\varepsilon > 0$ equation (13) has a pathwise unique strong solution. But from [10, Lemma 1] we have $\eta_\varepsilon(t) = \varphi(\xi_\varepsilon(t))$ or $\xi_\varepsilon(t) = \kappa(\eta_\varepsilon(t))$ so for every fixed $\varepsilon > 0$ equation (1) also has a pathwise unique strong solution.

Theorem 1 is proved.

Equation (13) corresponds with next Cauchy problem:

$$z'(t) = \widetilde{b}(z(t)), \quad z(0) = 0. \tag{14}$$

7

By the lemma 1 for the equation (14) we have next result: every solution of this problem is one of two following types

$$z_\lambda(t) = \widetilde{H}^{-1}\big((t-\lambda)^+\big), \quad \lambda \geq 0,$$

$$z_\mu(t) = \widetilde{K}^{-1}\big(-(t-\mu)^+\big), \quad \mu \geq 0,$$

where $\widetilde{H}(x) = \int_0^x \dfrac{1}{\widetilde{b}(y)} dy$ for $x \geq 0$; $\widetilde{K}(x) = \int_x^0 \dfrac{1}{\widetilde{b}(y)} dy$ for $x \leq 0$.

It is clear that $\widetilde{H}^{-1}(t)$ and $\widetilde{K}^{-1}(-t)$ are the solutions which first leave $[-r, r]$. This functions are called extremal (upper and lower respectively) and denote by $\overline{z}(t)$, $\underline{z}(t)$.

It is clear that strictly positive and strictly increasing for $t > 0$ functions $\overline{y}(t)$, $\overline{z}(t)$ are solutions of problem (2) and problem (14) respectively. Similarly, strictly negative and strictly decreasing for $t > 0$ functions $\underline{y}(t)$, $\underline{z}(t)$ are solutions of problem (2) and problem (14) respectively. Let's prove connection between extremal solutions of problem (2) and problem (14).

Lemma 3. $\overline{y}(t) = \kappa(\overline{z}(t)), \; \underline{y}(t) = \kappa(\underline{z}(t))$.

Proof. Let's prove lemma for the function $\overline{y}(t)$, for the function $\underline{y}(t)$ it be analogously. Let the function $y(t)$ be any solution of equation (2). Consider the function $z(t) = \varphi(y(t))$. Functions from the set $\mathfrak{R}$ don't change their signs, so we get from equation (2)

$$z(t) = \frac{y(t)}{1 + \beta \operatorname{sgn} y(t)} = \frac{1}{1 + \beta \operatorname{sgn} y(t)} \int_0^t b(\kappa(\varphi(y(s)))) ds =$$

$$= \int_0^t \frac{b(\kappa(\varphi(y(s))))}{1 + \beta \operatorname{sgn} \varphi(y(s))} ds = \int_0^t \widetilde{b}(z(s)) ds.$$

Thus the function $z(t) = \varphi(y(t))$ is a solution of equation (14). The function $\varphi(x)$ is strictly increasing and we have

$$z(t) = \varphi(y(t)) \leq \varphi(\overline{y}(t)) = \overline{z}(t).$$

Lemma 3 is proved.

Denote by ν_ε measures generated by the processes $\eta_\varepsilon(\cdot)$ on the space $(C[0, \infty), \mathfrak{B})$.

Proof of theorem 2. From conditions of the theorem 2 it implies that the coefficients of the process $\eta_\varepsilon(t)$ satisfy conditions [12, theorem 4.1]. Therefore

8

the measures $\{\nu_\varepsilon\}$ weakly converge to the measure ν concentrated on $\overline{z}$ and $\underline{z}$, i.e. for any continuous bounded functional F, defined on the space $\mathbb{C}[0,\infty)$, the next equality is valid:

$$\lim_{\varepsilon \to 0} \int_{\mathbb{C}[0,\infty)} F(f)\nu_\varepsilon(df) = \int_{\mathbb{C}[0,\infty)} F(f)\nu(df).$$

Moreover

$$\lim_{\varepsilon \to 0} \int_{\mathbb{C}[0,\infty)} F(f)\nu_\varepsilon(df) = \widetilde{\Gamma}F(\overline{z}) + \left(1 - \widetilde{\Gamma}\right)F(\underline{z}), \qquad (15)$$

The limit measure ν is given by the right-hand side of equality (15); $\overline{z}, \underline{z}$ are extreme solutions of the problem (14), and the value of $\widetilde{\Gamma}$ for functions $\widetilde{b}(t)$, $\widetilde{\sigma}(t)$ is defined by [12, formula (2.11)].

If we substitute in [12, formula (2.11)] functions $\widetilde{b}(t)$, $\widetilde{\sigma}(t)$, we get $\widetilde{\Gamma} = \Gamma$, where value of Γ is defined in (7).

Further from the definition of a measure generated by the process we have:

$$\mu_\varepsilon\{A\} = \mathsf{P}\{\xi_\varepsilon(\cdot) \in A\} = \mathsf{P}\{\kappa(\eta_\varepsilon(\cdot)) \in A\} = \mathsf{P}\{\eta_\varepsilon(\cdot) \in \varphi(A)\} = \nu_\varepsilon\{\varphi(A)\}.$$

From (15) and lemma 3 we can get

$$\nu(\varphi(A)) = \Gamma I_{\{\overline{z}(\cdot) \in \varphi(A)\}} + (1 - \Gamma)I_{\{\underline{z}(\cdot) \in \varphi(A)\}} =$$

$$= \Gamma I_{\{\kappa(\overline{z}(\cdot)) \in A\}} + (1 - \Gamma)I_{\{\kappa(\underline{z}(\cdot)) \in A\}} = \Gamma I_{\{\overline{y}(\cdot) \in A\}} + (1 - \Gamma)I_{\{\underline{y}(\cdot) \in A\}} = \mu(A),$$

where the measure μ is defined by the right-hand side of equality (11).

So for any continuous bounded functional F, defined on the space $\mathbb{C}[0,\infty)$, we have

$$\lim_{\varepsilon \to 0} \int_{\mathbb{C}[0,\infty)} F(y)\mu_\varepsilon\{dy\} = \lim_{\varepsilon \to 0} \int_{\mathbb{C}[0,\infty)} F(y)\nu_\varepsilon\{\varphi(dy)\} =$$

$$= \lim_{\varepsilon \to 0} \int_{\mathbb{C}[0,\infty)} F(\kappa(y))\nu_\varepsilon\{dy\} = \int_{\mathbb{C}[0,\infty)} F(\kappa(y))\nu\{dy\} =$$

$$= \int_{\mathbb{C}[0,\infty)} F(y)\nu\{\varphi(dy)\} = \int_{\mathbb{C}[0,\infty)} F(y)\mu\{dy\}.$$

Theorem 2 is proved.

9

Proof of theorem 3. Let's consider the cases A_2 or A_4, cases A_3 or A_5 one can consider by analogy. Similarly to the proof of theorem 2 initially for the process $\eta_\varepsilon(t)$ by [12, theorem 4.3] (with appropriate modifications) prove convergence of measures generated by the solutions of the equation (13) to the measure that is concentrated on the upper extreme solution of the corresponding Cauchy problem (14) and then return to the process $\xi_\varepsilon(t)$.

Theorem 3 is proved.

5 Examples.

Example 1. Let in the equation (1) coefficients have the form

$$
b(x) = \begin{cases} x^{\alpha_1}, & x \geq 0, \\ -C|x|^{\alpha_2}, & x \leq 0, \end{cases}
$$

$$
\sigma(x) = \begin{cases} \sigma_1, & x \geq 0, \\ \sigma_2, & x < 0, \end{cases}
$$

with constants $C > 0$, $\sigma_i > 0$, $0 < \alpha_i \leq 1$, $i = 1, 2$.

If $\alpha_1 < 1, \alpha_2 = 1$, then the first integral in (4) converges, and second one diverges, so we have the case A_2 and $\underline{y}(t) = 0$ by the lemma 1. Similarly, if $\alpha_1 = 1, \alpha_2 < 1$, then $\overline{y}(t) = 0$.

If $0 < \alpha_1, \alpha_2 < 1$, then we have the case A_1 and

$$
L(x) = \begin{cases} \dfrac{x^{\alpha_1+1}}{\sigma_1^2(\alpha_1 + 1)}, & x \geq 0, \\[3mm] -\dfrac{C|x|^{\alpha_2+1}}{\sigma_2^2(\alpha_2 + 1)}, & x < 0. \end{cases}
$$

Then conditions (9) and (10) are hold with the constants $\gamma = 0$, $d = \frac{1}{\sigma_1^2(\alpha_1+1)}$, $\delta = \alpha_1 + 1$; $\theta = 0, k = \frac{C}{\sigma_2^2(\alpha_2+1)}$, $\mu = \alpha_2 + 1$, which means that the conditions of theorem 2 take place. We have:
1. If $\alpha_1 = \alpha_2 = \alpha < 1$, then

$$
\Gamma = \frac{1}{1 + \frac{1-\beta}{1+\beta}\left(\dfrac{C\sigma_1^2}{\sigma_2^2}\right)^{\frac{1}{\alpha+1}}}.
$$

10

2. If $\alpha_1 < \alpha_2 \leq 1$, then $\Gamma = 1$.

3. If $\alpha_2 < \alpha_1 \leq 1$, then $\Gamma = 0$.

From the theorems 2, 3 we have that the limit measure is concentrated with weight Γ on the upper extremal solution and with weight $1 - \Gamma$ on the lower extremal solution of the corresponding Cauchy problem.

Example 2. Let in the equation (1) coefficient $\sigma(x)$ be such as in the example 1 and the drift coefficient be

$$
b(x) = \begin{cases} x^\alpha \left(|\ln x| + 1 \right), & x > 0, \\ -|x|^\alpha, & x \leq 0, \end{cases} \quad 0 < \alpha < 1.
$$

Then the condition A_1 and the conditions of the theorem 2 are hold. Value (7) can be calculated by the lemma 2, because conditions (9) and (10) take place with constants $\gamma = -1, d = \frac{1}{\sigma_1^2(\alpha+1)}, \delta = \alpha + 1; \quad \theta = 0, k = \frac{1}{\sigma_2^2(\alpha+1)}, \mu = \alpha + 1$.

According to the lemma 2, we have $\Gamma = 1$ and limit measure is concentrated on the upper extremal solution of the Cauchy problem (2).

Example 3. Let coefficients of the equation (1) have such form:

$$
b(x) = \begin{cases} x^\alpha, & x \geq 0, \\ -|x|^\alpha, & x \leq 0, \end{cases} \quad 0 < \alpha < 1.
$$

$$
\sigma(x) = \begin{cases} 2 - \cos x, & x \geq 0, \\ 2 + \cos x, & x < 0. \end{cases}
$$

In this case, the condition A_1 and other conditions of the theorem 2 are hold, thus there is a weak convergence.

According to the lemma 2 $\Gamma = \left(1 + \frac{1-\beta}{1+\beta} 9^{\frac{1}{\alpha+1}} \right)^{-1}$. So limit measure is concentrated with weight Γ on the upper extreme solution and with weight $1 - \Gamma$ on the lower extremal solution of the corresponding Cauchy problem.

References

[1] D.W. Stroock and S.R.S. Varadhan, *Multidimensional Diffusion Processes*, Springer-Verlag, Berlin, Heidelberg, New York, 1979.

[2] N. Ikeda and S. Watanabe, *Stochastic Differential Equations and Diffusion Processes*, North-Holland and Kodansha, Amsterdam, Oxford, New York, Tokyo. 1981.

[3] P. Baldi, *Petites perturbations d'un phenomene Peano*, Annales scientifiques de l'Universite de Clermont-Ferrand 2 **71** 1982, № 20, 41–52.

[4] P. Baldi and R. Bafico, *Small Random Perturbations of Peano Phenomena*, Stochastics **6** 1982, 279–292.

[5] A.Yu. Veretennikov, *Approximation of ordinary differential equations by stochastic ones* (Russian), Mat. Zametki **33** 1983, № 6, 929—932

[6] M.I. Freidlin and A.D. Wentzell, *Random perturbation of dynamical systems*, Springer-Verlag, New York, 1984.

[7] H.J. Engelbert and W. Schmidt, *Strong Markov continuous local martingales and solutions of one-dimensional stochastic differential equations, III*, Math. Nachr. **151** 1991, issue 1, 149–197.

[8] M. Gradinaru, S. Herrmann, B. Roynette, *A singular large deviations phenomenon*, Ann. Inst. H. Poincare Probab. Statist. **37** 2001, № 5, 555—580.

[9] S.Ya. Makhno, *A limit theorem for stochastic equations with the local time*, Theory of Probability and Mathematical Statistics, **64** 2001, 123–127.

[10] S.Ya. Makhno, *A limit theorem for one-dimensional stochastic equations*, Theory Probab. Appl. **48** 2003, 164–169.

[11] R. Buckdahn, M. Quincampoix, Y. Ouknine, *On limiting values of stochastic differential equations with small noise intensity tending to zero*, Bulletin des sciences mathematiques **133** 2009, 229–237.

[12] I.H. Krykun and S.Ya. Makhno, *Peano phenomenon for Ito's stochastic equations* (Russian), Ukrainian Mathematical Bulletin **10**, № 1, 2013, 87–109.